Last Time: Bases! Basis ⇝ Bases
(Basis: singular; Bases: plural)

Prop: Let V be a vector space w/ $B \subseteq V$.
The following are equivalent.

① B is a basis
② B is linearly independent and spans V.
③ B is a minimal spanning set.
(i.e. B spans V but no proper subset of B spans V
i.e. $\text{span}(B) = V$ but for all $b \in B$
$\text{span}(B \setminus \{b\}) \neq V$.)
④ B is a maximal independent set. $S \cup T = \{x : x \in S \text{ or } x \in T\}$
(i.e. B is independent in V but
for all $v \in V \setminus B$ we have $B \cup \{v\}$ is dep.)
⑤ Every vector in V can be expressed uniquely as a linear combination in B.

We'll prove ② ⟺ ⑤. Rest is left as an exercise.

pf: Let $B \subseteq V$ for some vector space V. Assume $B = \{b_1, b_2, \ldots, b_n\}$

②⇒⑤: Suppose B is lin. ind and spans V.

Let $u \in V$ be arbitrary. Because $\text{span}(B) = V$, we can write $u = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n$ for some $c_1, c_2, \ldots, c_n \in \mathbb{R}$. Assume there is another

linear combination $u = c_1' b_1 + c_2' b_2 + \cdots + c_n' b_n$. Hence

$$0_V = u - u = (c_1 b_1 + c_2 b_2 + \cdots + c_n b_n) - (c_1' b_1 + c_2' b_2 + \cdots + c_n' b_n)$$

$$= (c_1 - c_1') b_1 + (c_2 - c_2') b_2 + \cdots + (c_n - c_n') b_n .$$

Because B is linearly independent, we must have

$$c_1 - c_1' = c_2 - c_2' = \cdots = c_n - c_n' = 0$$

Thus $c_i - c_i' = 0$ for all i, so $c_i = c_i'$ for all i.

Hence these are the same linear combination of B, so we have a unique expression of u as a lin. comb.

③ $\Rightarrow$ ②: Assume every vector $u \in V$ can be expressed uniquely as a linear combination of vectors in B. Hence for any $u \in V$ there are coefficients $c_1, c_2, \ldots, c_n \in \mathbb{R}$ s.t. $u = c_1 b_1 + c_2 b_2 + \cdots + c_n b \in \mathrm{Span}(B)$.

Hence $V \subseteq \mathrm{Span}(B) \subseteq V$, so $\mathrm{Span}(B) = V$.

Note $0_V \in V$, so there is a unique linear combination of vectors in B yielding 0_V, namely

$0_V = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n$. On the other hand,

$0_V = \underbrace{0 b_1 + 0 b_2 + \cdots + 0 b_n}$, so EVERY 0_V linear combination in B is the trivial combination. Hence B is lin indep by definition. $\square$

Point: Given a vector $u \in V$ and two bases, B and B', we can compare their "representations" of u..
i.e. we can uniquely represent u as a vector in $\mathbb{R}^n$ for each of these bases, and compare...

Notation: $[u]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ when $u = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n$.

Ex: Let $B = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ and $u = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

B is a basis of $\mathbb{R}^2$ (check!).

To calculate $[u]_B$ we solve:

$$\left[\begin{array}{cc|c} 3 & -1 & 3 \\ 1 & 1 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 0 & -4 & -3 \\ 1 & 1 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & \frac{3}{4} \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 5/4 \\ 0 & 1 & 3/4 \end{array}\right]$$

$\therefore$ we've calculated coefficients $c_1 = \frac{5}{4}$ and $c_2 = \frac{3}{4}$ i.e.

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{5}{4}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{(check directly!)}$$

$$\therefore \quad [u]_B = \begin{bmatrix} 5/4 \\ 3/4 \end{bmatrix}.$$

Let $B' = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. Now to compute $[u]_{B'}$:

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \quad \text{so} \quad [u]_{B'} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \quad ▨$$

Note: $[u]_B \neq [u]_{B'}$ ...

Ex: In $\mathbb{R}^n$, $\mathcal{E}_n = \{e_1, e_2, \ldots, e_n\}$ and every vector $u \in \mathbb{R}^n$ has $[u]_{\mathcal{E}_n} = u$ ▨

Idea: create new bases from old ones...

Lem (Steinitz Exchange Lemma): If $B = \{b_1, b_2, \dots, b_n\}$ is a basis of vector space V and $u = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n$ has $c_i \neq 0$, then $B \setminus \{b_i\} \cup \{u\}$ is a basis of V.

pf: Let V be a vector space and $B \subseteq V$ be a basis.

Assume $u = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n$ with $c_i \neq 0$.

(WTS: $B \setminus \{b_i\} \cup \{u\} = \{b_1, b_2, \dots, b_{i-1}, u, b_{i+1}, \dots, b_n\}$ is a basis)

Let $w \in V$ be arbitrary. We may express

$$w = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \quad \text{for some } a_1, \dots, a_n \in \mathbb{R}.$$

Note $b_i = \frac{1}{c_i}\left(u - c_1 b_1 - c_2 b_2 - \cdots - c_{i-1} b_{i-1} - c_{i+1} b_{i+1} - \cdots c_n b_n\right)$

In particular,

$$w = a_1 b_1 + a_2 b_2 + \cdots + a_i b_i + \cdots + a_n b_n$$

$$= a_1 b_1 + a_2 b_2 + \cdots + a_i \left(\frac{1}{c_i} u - \frac{c_1}{c_i} b_1 - \cdots - \frac{c_{i-1}}{c_i} b_{i-1} - \frac{c_{i+1}}{c_i} b_{i+1} - \cdots - \frac{c_n}{c_i} b_n\right) + \cdots + a_n b_n$$

$$= \left(a_1 - \frac{a_i c_1}{c_i}\right) b_1 + \left(a_2 - \frac{a_i c_2}{c_i}\right) b_2 + \cdots + \frac{a_i}{c_i} u + \cdots + \left(a_n - \frac{a_i c_n}{c_i}\right) b_n$$

Hence $w \in \operatorname{span}(B \setminus \{b_i\} \cup \{u\})$; as $w \in V$ was arbitrary, so $\operatorname{span}(B \setminus \{b_i\} \cup \{u\}) = V$.

To see $B \setminus \{b_i\} \cup \{u\}$ is lin indep, suppose

$$0_V = a_1 b_1 + a_2 b_2 + \cdots + a_i u + \cdots + a_n b_n.$$

(First we'll show $a_i = 0$). Replacing $u = c_1 b_1 + \cdots + c_n b_n$,

$$0_V = a_1b_1 + a_2b_2 + \underbrace{\cdots +}_{\text{no } b_i} a_i(c_1b_1 + c_2b_2 + \cdots + c_nb_n) + \underbrace{\cdots +}_{\text{no } b_i} a_nb_n.$$
$$= (a_1 + a_ic_1)b_1 + (a_2 + a_ic_2)b_2 + \cdots + a_ic_ib_i + \cdots + (a_n + a_ic_n)b_n$$

As B is linearly independent, we have

$$[a_j + a_ic_j] = 0 \text{ for all } j \neq i \text{ and } \underline{a_ic_i = 0}$$

Because $a_ic_i = 0$, we see either $a_i = 0$ or $[c_i = 0]$.

But $c_i \neq 0$ by assumption, so $a_i = 0$. On the other hand,

$0 = a_j + a_ic_j = a_j + 0c_j = a_j$, so all the coefficients

in $a_1b_1 + a_2b_2 + \cdots + a_iu + \cdots a_nb_n = 0_V$ must be

$a_j = 0$; Thus $B \setminus \{b_i\} \cup \{u\}$ is lin. indep.

Hence $B \setminus \{b_i\} \cup \{u\}$ is lin. indep and spanning, so it is a basis! $\boxtimes$

<u>Point:</u> Given $u \in V$ and basis B of V, we can exchange u for any vector in B w/ coeff. $c \neq 0$ in the representation of u w.r.t. B.

<u>Cor 1:</u> Given bases A and B of V, and vector $a \in A$, there is a vector $b \in B$ such that $A \setminus \{a\} \cup \{b\}$ is a basis of V.

<u>Sketch:</u> a has a representation $[a]_B$ w/ at least one nonzero coeff, so choose any $b \in B$ w/ $[a]_B$ has nonzero component for b. $\boxtimes$

Cor 2: If V has a finite basis, then every basis has the same number of elements.

Sketch: Given bases A and B of V and a finite basis F of V, we proceed as follows. Take $f \in F \setminus A$. We can find $a \in A$ s.t. $F \setminus \{f\} \cup \{a\}$ is a basis. Do so until you remove all elements of $F \setminus A$.

The result is a basis contained in A. Thus, the result is itself A. At each step, the number of elements in our basis remains the same. $\boxtimes$